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On the Uniform Modulus of Continuity of Certain Discrete Approximation Operators

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For a certain class of discrete approximation operators B_n^f defined on an interval I and including, e.g., the Bernstein polynomials, we prove that for all $f \in C(I)$, the ordinary moduli of continuity of B_n^f and f satisfy

$$\omega(B_n^f; h) \leqslant c\omega(f; h), \qquad n = 1, 2, ..., 0 < h < \infty,$$

with a universal constant $c > 0$. A similar result is shown to hold for a different modulus of continuity which is suitable for functions of polynomial growth on unbounded intervals. Some special operators are discussed in this connection.

1. INTRODUCTION

When investigating the approximation quality of certain smooth approximations of a Wiener process, it is important to control (uniformly) the modulus of continuity of the smooth process [15, 16]. So far kernel approximation operators with L_1-kernels were studied, where the kernels have bounded support, say $[-1, 1]$, and integrate to one. For a function $f \in C(\mathbb{R})$, a kernel approximation operator is defined by

$$K_b^f(x) = \frac{1}{b} \int K\left(\frac{x-t}{b}\right) f(t)\, dt, \qquad b > 0 \text{ and } x \in [0, 1].$$

Using the substitution rule, the modulus of continuity of K_b^f may be estimated as follows. If $\omega(f; \cdot)$, $\omega(K_b^f; \cdot)$, respectively, denote the moduli of continuity [12, p. 51] of f on $[-1, 2]$ and of K_b^f on $[0, 1]$, then

$$\omega(K_b^f; h) \leqslant c\omega(f; h), \qquad \text{for all } h \geqslant 0 \text{ and } \textit{all } b \in (0, 1], \tag{1}$$

$c > 0$ being a constant (e.g., $c = \|K\|_1$). For discrete approximation operators an inequality like (1) is not so immediate. Actually, in [4,

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Sect. 3.2] Bloom and Elliot say that even for Bernstein polynomials the optimal (uniform) estimate (like (1)) of their modulus of continuity is not known (e.g., when $f \in \mathrm{Lip}(\mu; C[0, 1])$, see formula (3.5) of [4]).

It is the aim of this paper to derive an inequality like (1) for a certain class of discrete (and essentially positive) approximation operators, which include, e.g., the Bernstein polynomials, the Szasz–Mirakjan operators [13, 17], or the Baskakov operators [2]. To be more precise, we consider on a given interval I for functions $f \in C(I)$ the approximating functions

$$B_n^f(x) = \sum_{j \in J_n} f(\xi_{jn})\, p_{jn}(x), \qquad \text{for } x \in I,\ n \in \mathbb{N}, \text{ with}$$
$$J_n \subseteq \mathbb{Z},\ \xi_{jn} \in I \text{ for } n \in \mathbb{N},\ j \in J_n. \tag{2}$$

If the "weights" $p_{jn}(x)$ satisfy the assumptions (the most important case of our main result below)

$$p_{jn}(x) \geqslant 0, \quad \sum_{j \in J_n} p_{jn}(x) \equiv 1, \quad 0 < \mu_{2n}(x) = \sum_{j \in J_n} (\xi_{jn} - x)^2\, p_{jn}(x) < \infty,$$
$$\text{and } p_{jn}(x) \in C_1(I) \text{ with } p_{jn}'(x)\, \mu_{2n}(x) = p_{jn}(x)(\xi_{jn} - x), \text{ for } x \in \mathring{I}, \tag{3}$$

then we shall prove that

$$\omega(B_n^f; h) \leqslant c\omega(f; h), \quad \text{for all } h \geqslant 0 \text{ and } all\ n \in \mathbb{N} \text{ with } c = 4. \tag{1'}$$

(If nothing specific is said, the modulus of continuity refers to the interval I, i.e., where f and B_n^f are defined.) Observe also, that up to the constant c the inequality (1′) cannot be improved if $B_n^f \to f$ as $n \to \infty$, and in that case we must have $c \geqslant 1$. Approximation operators satisfying (3) occur, e.g., if the $\{p_{jn}(x)\}$ are the n-fold convolutions of a lattice distribution $\{p_{j1}(x)\}$ which satisfy (3) (for $n = 1$) as discussed in [7, 9, 18, 19]. A rather complete bibliography concerning approximation operators of Bernstein type was provided by Gonska and Meier [10].

In order to obtain an inequality of type (1′) one could try to estimate $B_n^{f\prime}(x)$. But even in the *case of Bernstein polynomials*, i.e.,

$$\xi_{jn} = j/n,\ p_{jn}(x) = \binom{n}{j} x^j (1-x)^{n-j},\ I = [0, 1], \qquad \mu_{2n}(x) = x(1-x)/n, \tag{4}$$

neither the standard inequality [11, Sect. 1.4] $|B_n^{f\prime}(x)| \leqslant n\omega(f; 1/n)$ nor the inequality $|B_n^{f\prime}(x)| \leqslant ((1/h) + \mu_{2n}(x)^{-1/2})\, \omega(f; h)$ for $h > 0$, $x \in (0, 1)$ (which can be derived from [3, p. 695]), nor a related result by Ditzian [5] leads to an inequality of type (1′). But, if the inequality on $B_n^{f\prime}(x)$ of [3, p. 695] together with the inequality

$$\left|\int_x^y \mu_{2n}(t)^{-1/2}\,dt\right| \leqslant 2\,|y-x|\,(\min\{\mu_{2n}(x)^{-1/2}, \mu_{2n}(y)^{-1/2}\}) \tag{5}$$

giving: $|B_n^f(x) - B_n^f(y)|$

$$\leqslant |y-x| \cdot \omega(f;h)\left(\frac{1}{h} + 2\cdot(\min\{\mu_{2n}(x)^{-1/2}, \mu_{2n}(y)^{-1/2}\})\right);$$

is combined appropriately with the approximation quality of the Bernstein polynomials, namely with

$$|B_n^f(x) - f(x)| \leqslant \left(1 + \frac{1}{\alpha^2}\right)\omega(f;\alpha\mu_{2n}(x)^{+1/2})$$

(compare the proof of Popoviciu's Theorem 1.6.1 in [11] or [14]) for $\alpha > 0$, $x \in (0, 1)$, then one gets (1′) with a constant $c < 4$ (e.g., $c = 669/169$ with $\alpha = 13/9$). Observe that this derivation requires (5) essentially, which does not hold for the Baskakov operators, e.g. Our proof below uses neither the approximation quality nor any estimate of $\mu_{2n}(x)$ like (5).

Besides our main result (Theorem 1) we derive similar estimates for a modified modulus of continuity in case that the interval I is infinite (Theorem 2). Moreover, a detailed discussion of special operators can be found in Section 3.

2. Results

THEOREM 1. *Let there be given an operator of type* (2), *such that the weights* $p_{jn}(x)$ *satisfy*

$$\sum_{j\in J_n} p_{jn}(x) \equiv s_n, \qquad \sum_{j\in J_n} |p_{jn}(x)| \leqslant c_1,$$

$$|\mu_{2n}|\,(x) := \sum_{j\in J_n} (\xi_{jn} - x)^2\,|p_{jn}(x)| < \infty,$$

and $p_{jn} \in C_1(I)$ *with* $\sum_{j\in J_n} |(\xi_{jn} - x)\,p'_{jn}(x)| \leqslant c_2$, *for all*

$$x \in I,\ n \in \mathbb{N}. \tag{3′}$$

Then, for any $f \in C(I)$, *we have*

$$\omega(B_n^f; h) \leqslant c\omega(f; h), \textit{ for all } h \geqslant 0, \textit{ and all } n \in \mathbb{N} \textit{ with}$$

$$c = 2(c_1 + c_2), \tag{1″}$$

where $\omega(B_n^f; \cdot)$ resp. $\omega(f; \cdot)$ denote the moduli of continuity of B_n^f resp. f on I.

Remark 1. (i) Observe that the last inequality of (3′) follows from

$$|p'_{jn}(x)| \leqslant c_2 |(\xi_{jn} - x)\, p_{jn}(x)|/|\mu_{2n}|(x), \qquad \text{for } j \in J_n, \text{ if } |\mu_{2n}|\,(x) > 0, \quad (6)$$

and this holds with $c_2 = 1$ under the assumptions (3).

(ii) The constant c (which is $=4$ in the most important case (3), when $c_1 = c_2 = 1$) can certainly be improved for special operators (e.g., for the Bernstein operators as discussed in the introduction). But it is not clear what the best possible constant is. Actually, if (3) holds and if, e.g., $I \supseteqq [0, a]$, $0 \in J_n$, $p_{0n}(0) = 1$ with $p_{0n}(h) \to 0$, and $\sum_{\xi_{jn} \geqslant h} p_{jn}(h) \to 1/2$ as $n \to \infty$ for $0 < h < a$ (which may be derived, e.g., from the central limit theorem), then the constant c must be $\geqslant 3/2$. Since, consider functions $g = g(x; \varepsilon)$ given by $g(0) = 1$, $g \equiv 0$ on $[\varepsilon, h - \varepsilon]$, $g(x) \equiv -1$ for $x \geqslant h$, and g linear and continuous on $[0, \varepsilon]$ resp. $[h - \varepsilon, h]$, and let $h(\varepsilon) = h - \varepsilon$. Then

$$\varliminf_{n \to \infty} \sup_{f \in C(I)} \omega(B_n^f; h)/\omega(f; h) \geqslant \lim_{n \to \infty} \lim_{\varepsilon \to 0+} (B_n^g(0) - B_n^g(h(\varepsilon))) = 3/2,$$

for all $0 < h < |I|$.

Proof. By (3′) our assertion follows immediately from the following inequality, which we shall derive, i.e.,

$$|B_n^f(x) - B_n^f(y)| \leqslant \omega(f; h) \left\{ \sum_{j \in J_n} |p_{jn}(x) - p_{jn}(y)| + \frac{2}{h} \left| \int_x^y \sum_{|\xi_{jn} - t| \geqslant h/2} |(\xi_{jn} - t)\, p'_{jn}(t)|\, dt \right| \right\},$$

$$\text{for all } n \in \mathbb{N},\ h > 0, \text{ and all } x, y \in I \text{ with } |x - y| \leqslant h. \quad (7)$$

Now, let $x, y \in I$ be given, $n \in \mathbb{N}$, $h > 0$ with $0 < y - x \leqslant h$, and put $x_0 = (x + y)/2$. For the proof of (7) we use a well known property of the modulus of continuity [11, p. 20], i.e.,

$$|f(t) - f(\tau)| \leqslant \left(1 + \left[\frac{|t - \tau|}{h}\right]\right) \omega(f; h), \qquad \text{for} \quad h > 0,\ t, \tau \in I, \quad (8)$$

and the inequality

$$|\xi - x_0|/|\xi - t| \leqslant 2, \qquad \text{for } t \in [x, y] \text{ whenever } |\xi - x_0| \geqslant h \geqslant y - x. \quad (9)$$

Then, since $\sum_{j\in J_n} p_{jn}(x)$ is constant by (3′), we obtain

$$|B_n^f(x)-B_n^f(y)| = \left|\sum_{j\in J_n}(f(\xi_{jn})-f(x_0))(p_{jn}(y)-p_{jn}(x))\right| \qquad \text{(by (8))}$$

$$\leqslant \omega(f;h)\sum_{j\in J_n}\left\{1+\left[\frac{|\xi_{jn}-x_0|}{h}\right]\right\}\left|\int_x^y p'_{jn}(t)\,dt\right|$$

$$\leqslant \omega(f;h)\left\{\sum_{j\in J_n}|p_{jn}(x)-p_{jn}(y)| + \frac{1}{h}\int_x^y \sum_{|\xi_{jn}-x_0|\geqslant h}|(\xi_{jn}-x_0)\,p'_{jn}(t)|\,dt\right\}$$

and this is $\leqslant$ the right-hand side of (7) by (9). ∎

Remark 2. Instead of estimating the summands of the right-hand side of (7) *uniformly* by $2c_1$ resp. $2c_2$, one can obtain the following nonuniform bounds using essentially the Cauchy–Schwarz inequality, provided that (3′) and (6) hold (compare [11] and (5)):

$$\sum_{j\in J_n}|p_{jn}(x)-p_{jn}(y)| \leqslant c_2\left|\int_x^y \frac{|\mu_{1n}|\,(t)}{|\mu_{2n}|\,(t)}\,dt\right| \leqslant c_2\sqrt{c_1}\left|\int_x^y \frac{dt}{\sqrt{|\mu_{2n}|\,(t)}}\right|,$$

for $x, y\in I$, $n\in\mathbb{N}$, and

$$\frac{2}{h}\left|\int_x^y \sum_{|\xi_{jn}-t|\geqslant h/2}|(\xi_{jn}-t)\,p'_{jn}(t)|\,dt\right| \leqslant c_2\left(\frac{2}{h}\right)^{\alpha-1}\left|\int_x^y \frac{|\mu_{\alpha n}|\,(t)}{|\mu_{2n}|\,(t)}\,dt\right|,$$

for $x, y\in I$, $h>0$, $n\in\mathbb{N}$, and $\alpha\geqslant 2$, where

$$|\mu_{\alpha n}|\,(t)=\sum_{j\in J_n}|\xi_{jn}-t|^{\alpha}\,|p_{jn}(t)|,$$

whenever it exists. Observe that the first inequality is "useful" only if $|y-x|$ is small compared to $n^{-1/2}$ (since $|\mu_{1n}|\,(t)/|\mu_{2n}|\,(t)\geqslant (|\mu_{2n}|\,(t)/|\mu_{4n}|\,(t))^{1/2}$ by Hölder's inequality, and this tends to ∞ in the most "relevant" examples); while the second inequality may be used if n is large compared to $1/h$ (since $|\mu_{\alpha n}|\,(t)/|\mu_{2n}|\,(t)$ tends to zero as $n\to\infty$ for $\alpha>2$ in "most" examples).

Of course, Theorem 1 is "useful" only if $\omega(f;h)<\infty$, which is satisfied for a compact interval I. But in case that the interval is infinite the assumption $|f(x)-f(y)|\leqslant\omega(f;|x-y|)<\infty$ for $x, y\in I$ implies that the growth of $|f(x)|$ at infinity is limited by $c\cdot|x|$. To overcome this restriction we choose a different modulus of continuity, which allows us to deal with functions of polynomial growth. Our modulus of continuity is essentially the same as the one which can be found in Achieser [1, p. 201]. In the following we use the assumption $f \in \tilde{C}_\sigma(I)$, which means

$$\sup_{t \in I} |f(t)|/(1+|t|)^\sigma < \infty, \qquad f \in C(I), \text{ where } \sigma \geq 0. \tag{10}$$

For those functions we consider the modulus of continuity Ω_σ, defined by

$$\Omega_\sigma(f; h) = \sup\{|f(t) - f(\tau)|/(1+|t|+|\tau|)^\sigma : t, \tau \in I, |t-\tau| \leq h\}. \tag{11}$$

We need a result analogous to (8), which reads for Ω_σ as follows:

$$|f(t) - f(\tau)| \leq \left(1 + \left[\frac{|t-\tau|}{h}\right]\right)(1+2|t|+2|\tau|)^\sigma \,\Omega_0(f; h),$$
$$\text{for} \quad t, \tau \in I, h > 0. \tag{12}$$

In order to verify this define $m := 1 + [|t-\tau|/h] \in \mathbb{N}$ for $t \leq \tau$ and let $t_\nu = t + \nu h$, $\nu = 0, ..., m-1$, $t_m = \tau$. Then, we obtain

$$|f(t) - f(\tau)| \leq \sum_{\nu=1}^{m} |f(t_\nu) - f(t_{\nu-1})| \left(\frac{1+|t_{\nu-1}|+|t_\nu|}{1+|t_{\nu-1}|+|t_\nu|}\right)^\sigma$$
$$\leq m\Omega_\sigma(f; h)(1+2|t|+2|\tau|)^\sigma$$

since $|t_m - t_{m-1}| \leq h$ and since $\Omega_\sigma(f; h)$ is increasing in h.

THEOREM 2. *Let there be given an operator of type* (2) *and a constant* $\sigma \geq 0$, *such that the weights* $p_{jn}(x)$ *satisfy* (3') *and*

$$|\tilde{\mu}_{\sigma,n}|\,(x) := \sum_{|\xi_{jn} - x| > 1} |\xi_{jn} - x|^{\sigma+1} |p'_{jn}(x)| \leq c_3(1+|x|)^\sigma,$$
for all $x \in \mathring{I}$, $n \in \mathbb{N}$ *with some constant* $c_3 = c_3(\sigma) > 0$. (13)

Then, for any $f \in \tilde{C}_\sigma(I)$, *we have*

$$\Omega_\sigma(B_n^f; h) \leq c\Omega_\sigma(f; h), \qquad \textit{for all } h \geq 0 \textit{ and all } n \in \mathbb{N}$$
$$\textit{with } c = c(\sigma) = 2(c_1 + c_2)\,5^\sigma + \frac{c_3}{2}\left(\frac{20}{3}\right)^{\sigma+1}. \tag{1'''}$$

Proof. We follow the lines of the proof of Theorem 1. So, let $x, y \in \mathring{I}$ be given, $n \in \mathbb{N}$, $h > 0$, with $0 < y - x \leq h$, and put $x_0 = (x+y)/2$. We consider $\Delta = |B_n^f(x) - B_n^f(y)|/(1+|x|+|y|)^\sigma$. Using (3') and (12) we get

$$\begin{aligned}\Delta &= \left|\sum_{j\in J_n}(f(\xi_{jn})-f(x_0))\frac{p_{jn}(y)-p_{jn}(x)}{(1+|x|+|y|)^\sigma}\right|\\ &\leqslant \Omega_\sigma(f;h)\sum_{j\in J_n}\left\{1+\left[\frac{|\xi_{jn}-x_0|}{h}\right]\right\}\left(\frac{1+2|\xi_{jn}|+2|x_0|}{1+|x|+|y|}\right)^\sigma\left|\int_x^y p'_{jn}(t)\,dt\right|\\ &= \Omega_\sigma(f;h)\left\{\sum_{|\xi_{jn}|\leqslant 2(1+|x|+|y|)}\cdots+\sum_{|\xi_{jn}|>2(1+|x|+|y|)}\cdots\right\}\\ &= \Omega_\sigma(f;h)\{I+II\}.\end{aligned}$$

For I observe that $1+2|\xi_{jn}|+2|x_0|\leqslant 5+4|x|+4|y|+2|x_0|\leqslant 5(1+|x|+|y|)$ (since $|x_0|\leqslant(|x|+|y|)/2$), hence

$$I\leqslant 5^\sigma\sum_{j\in J_n}\left\{1+\left[\frac{|\xi_{jn}-x_0|}{h}\right]\right\}\left|\int_x^y p'_{jn}(t)\,dt\right|\leqslant 5^\sigma\cdot 2(c_1+c_2)$$

by the proof of Theorem 1. The second term II can be treated as follows. For $|\xi_{jn}|>2(1+|x|+|y|)$ we have (with $h':=y-x\leqslant h$) $|\xi_{jn}-x_0|\geqslant |\xi_{jn}|-|x_0|>2(1+|x|+|y|)-|x_0|\geqslant\max\{2+3|x_0|,\ 3h'/2\}$; hence, $|\xi_{jn}|>2+4|x_0|$, $|\xi_{jn}-x_0|>\max\{2,h'\}$, $\frac{8}{3}<\frac{4}{3}(|\xi_{jn}|-4|x_0|)$, which implies $1+2|\xi_{jn}|+2|x_0|<2|\xi_{jn}|+2|x_0|+\frac{8}{3}<\frac{10}{3}(|\xi_{jn}|-|x_0|)\leqslant\frac{10}{3}|\xi_{jn}-x_0|$ and $1+|x|+|y|\geqslant 1+|t|$ for $x\leqslant t\leqslant y$. Altogether we obtain

$$II\leqslant\left(\frac{10}{3}\right)^\sigma\sum_{|\xi_{jn}-x_0|>\max\{2,h'\}}|\xi_{jn}-x_0|^\sigma\left\{1+\frac{|\xi_{jn}-x_0|}{h'}\right\}\int_x^y\frac{|p'_{jn}(t)|}{(1+|t|)^\sigma}\,dt$$

(observe $|\xi_{jn}-x_0|\geqslant 3h'/2$)

$$\begin{aligned}&\leqslant\frac{5}{3h'}\left(\frac{10}{3}\right)^\sigma\sum_{|\xi_{jn}-x_0|>\max\{2,h'\}}|\xi_{jn}-x_0|^{\sigma+1}\int_x^y\frac{|p'_{jn}(t)|}{(1+|t|)^\sigma}\,dt\\ &\leqslant\frac{10}{3h'}\left(\frac{20}{3}\right)^\sigma\int_x^y\sum_{|\xi_{jn}-x_0|>\max\{2,h'\}}|\xi_{jn}-t|^{\sigma+1}\frac{|p'_{jn}(t)|}{(1+|t|)^\sigma}\,dt\end{aligned}$$

using (3′) and (9) with h' instead of h. Since $|\xi_{jn}-x_0|>\max\{2,h'\}$ implies that $|\xi_{jn}-t|>1$ for $x\leqslant t\leqslant y$, we get $II\leqslant(c_3/2)(\frac{20}{3})^{\sigma+1}$ by (13), which proves Theorem 2. ∎

Remark 3. (i) Observe that (13) follows, e.g., from (6) and from

$$\sum_{|\xi_{jn}-x|>1}|\xi_{jn}-x|^{\sigma+2}\,|p_{jn}(x)|\leqslant\frac{c_3}{c_2}(1+|x|)^\sigma\,|\mu_{2n}|(x),\qquad(13')$$

which holds in our examples (i)–(iii) below.

(ii) Finally, we remark shortly how to handle approximation operators which do not have a constant sum of weights, as there are, e.g., the Favard operators (see next section). The proofs of Theorems 1 and 2 show immediately that under the same assumptions (*except* the constancy of the sum of weights) the following estimates (with the same constant c) hold:

$$\omega(B_n^f; h) \leqslant c\omega(f; h) + c^f\omega^*(h); \qquad \text{resp.} \tag{1*}$$

$$\Omega_\sigma(B_n^f; h) \leqslant c\Omega_\sigma(f; h) + c_\sigma^f\omega^*(h), \tag{1**}$$

where $c^f = \sup\{|f(x)| : x \in I\}$, $c_\sigma^f = \sup\{|f(x)/(1+|x|)^\sigma : x \in I\}$, and where $\omega^*(h)$ denotes the "uniform" modulus of continuity of the sum of the weights, i.e.,

$$s_n(x) := \sum_{j \in J_n} p_{jn}(x),$$

$$\omega^*(h) = \sup\{|s_n(x) - s_n(y)| : x, y \in I, |x-y| \leqslant h, n \in \mathbb{N}\}.$$

3. Discussion of Special Operators

Many positive approximation operators are based on probability distributions. Here we prove an auxiliary result (to derive assumption (13) of Theorem 2) for operators where the weights $p_{jn}(x)$ are the n-fold convolution of a discrete distribution on I. To be more precise, we assume that

> $\{p_{jn}(x)\} = \{p_{j1}(x)\}^{*n}$, $\xi_{jn} = j/n$, where $\{p_{j1}(x), x \in I\}$ is a lattice distribution concentrated on some set $J \subset \mathbb{Z} \cap I$ with $\sum (j-x)^{2N} p_{j1}(x) < \infty$ for all $x \in I$, where the sum converges uniformly on compact subsets of I for some $N \in \mathbb{N}$; and $p_{j1} \in C_1(I)$, $\mu_{21}(x)\, p'_{j1}(x) = p_{j1}(x)(j-x)$, $\mu_{21}(x) = \sum (j-x)^2 p_{j1}(x) > 0$ for $x \in \mathring{I}$. (14)

From these assumptions the following properties of the $p_{jn}(x)$ follow by [7].

> $\mu_{kn}(x) = \sum (\xi_{jn} - x)^k p_{jn}(x)$ exists for all $x \in I$, $k = 0, ..., 2N$, and the sum converges uniformly and absolutely on compact subsets of I; $p_{jn}(x) \geqslant 0$, $\mu_{0n}(x) \equiv 1$, $\mu_{1n}(x) \equiv 0$, $\mu_{2n}(x) = \mu_{21}(x)/n$; $p_{jn} \in C_1(I)$ with $\mu_{2n}(x)\, p'_{jn}(x) = (\xi_{jn} - x)\, p_{jn}(x)$; $\mu_{kn}(x) = \sum_{\nu=1}^{[k/2]} a_\nu(x)\, n^{\nu-k}$, $a_\nu \in \mathbb{R}[\mu_{21}(x), ..., \mu_{k1}(x)]$, $\nu = 1, ..., [k/2]$, $k = 2, ..., N$. (15)

A direct consequence of these results is

$\mu_{kn} \in C_1(I)$ with $\mu_{k+1,n}(x) = (\mu'_{kn}(x) + k\mu_{k-1,n}(x))\,\mu_{21}(x)/n$ for $k = 1, ..., 2N-1$; and $\mu_{kn}(x)$ is a polynomial of degree $\leqslant [k/2]$ resp. k whenever $\mu_{21}(x)$ is a polynomial of degree $= 1$ resp. $= 2$. (16)

LEMMA. *If* (14) *holds, then* (3) (*and hence* (3′) *with* $c_1 = c_2 = 1$) *holds. Moreover, if in addition* $\mu_{21}(x)$ *is a polynomial of degree* 1 *or* 2, *then* (13) *holds for all* $0 \leqslant \sigma \leqslant 2N-2$.

Proof. Assume $\sigma \in [0, 2m]$, $m \in \{0, 1, ..., N-1\}$, and let $K > 0$ be sufficiently large. Then, by (15) and Hölder's inequality,

$$\sum |\xi_{jn} - x|^{\sigma+1}\,|p'_{jn}(x)| = \sum |\xi_{jn} - x|^{\sigma+2}\,p_{jn}(x)/\mu_{2n}(x)$$

$$\leqslant \frac{1}{\mu_{2n}(x)}\,(\mu_{2m+2,n}(x))^{(2+\sigma)/(2m+2)}$$

and this is $\leqslant c_3(1+|x|)^\sigma$ for $|x| \geqslant K$ if $\mu_{21}(x)$ has degree 2, and $\leqslant c_3(1+|x|)^{\sigma/2} \leqslant c_3(1+|x|)^\sigma$ for $|x| \geqslant K$ if $\mu_{21}(x)$ has degree 1 by (16) and (15). Moreover, we have

$$\sum_{|\xi_{jn} - x| \geqslant 1} |\xi_{jn} - x|^{\sigma+1}\,|p'_{jn}(x)| \leqslant \mu_{2m+2,n}(x)/\mu_{2n}(x) \leqslant c_3$$

on the compact interval $[-K, K]$ by (15) and (16). ∎

Remark 4. Observe that the constant $c_3 = c_3(\sigma)$ occurring in (13) may be estimated explicitly in our examples (i)–(iv) below, but we omit this rather tedious calculation.

EXAMPLES. (i) *Bernstein polynomials* [11], defined by (4). Now, (14) holds with $I = [0, 1]$ and so Theorem 1 applies with $c = 4$.

(ii) *Szasz–Mirakjan operators* [13, 17], defined by $\xi_{jn} = j/n$, $p_{jn}(x) = e^{-nx}(nx)^j/j!$, $x \in I = [0, \infty)$, $j \in J_n = \mathbb{N}_0$ with $\mu_{2n}(x) = x/n$ ($\{p_{j1}(x)\}$ is the Poisson distribution with parameter $\lambda = x$). Again (14) holds and Theorem 1 applies with $c = 4$. Moreover, by the Lemma, Theorem 2 applies *for all* $\sigma \geqslant 0$ with $c = 4 \cdot 5^\sigma + \frac{1}{2}c_3(\sigma)(20/3)^{\sigma+1}$ and with $c_3(\sigma)$ according to (13) (compare Remark 4).

(iii) *Baskakov operators* [2], defined by

$$\xi_{jn} = j/n, \qquad p_{jn}(x) = \binom{n+j-1}{j} x^j (1+x)^{-n-j},$$

$$x \in I = [0, \infty),\ j \in J_0 = \mathbb{N}_0$$

with $\mu_{2n}(x)=x(1+x)/n$ ($\{p_{j1}(x)\}$ is the geometric distribution with parameter $q=x/(1+x)$). As in example (ii) Theorem 1 applies with $c=4$ and Theorem 2 applies for all $\sigma\geqslant 0$ with the corresponding c. By a change of variable $x\to y/(1+y)$ in the Baskakov operator we obtain the so-called *Meyer–König–Zeller operator* with $\xi_{jn}=j/(j+n)$ [18, 20].

(iv) *Generalized Favard operators* [8], defined by

$$\xi_{jn}=j/n, \qquad p_{jn}(x)=\frac{1}{n\sigma_n\sqrt{2\pi}}\exp\left(-\frac{(j-nx)^2}{2n^2\sigma_n^2}\right)$$

with $\sigma_n>0$, $x\in I=\mathrm{R}$, $j\in J_n=\mathbb{Z}$. ($\sigma_n^2=\lambda/2n$, $\lambda>0$ corresponds to the "classical" Favard operators [6].) Now, $\sum p_{jn}(x)$ is not constant so that Theorems 1 and 2 can be applied only in connection with Remark 3. By [8, formula (1.3)] we have

$$\sum_{j=-\infty}^{\infty} p_{jn}(x)=1+r_n(x) \quad \text{with}$$

$$r_n(x)=2\sum_{\nu=1}^{\infty}\cos(2\pi\nu n x)\exp(-2\pi^2\nu^2n^2\sigma_n^2).$$

Moreover, we have

$$p'_{jn}(x)=(\xi_{jn}-x)\,p_{jn}(x)/\sigma_n^2,$$

$$\mu_{k+1,n}(x)=\sigma_n^2(\mu'_{kn}(x)+k\mu_{k-1,n}(x)), \qquad k=1,2,...$$

(compare (15) and (16)) and this yields

$\mu_{0n}(x)=1+r_n(x)$, $\mu_{1n}(x)=\sigma_n^2r'_n(x)$, $\mu_{2n}(x)=\sigma_n^2(1+r_n(x)+\sigma_n^2r''_n(x))$, and $\mu_{2k,n}(x)\leqslant d_k\sigma_n^{2k}(1+\sum_{\nu=0}^{k}\sigma_n^{2\nu}|r_n^{(2\nu)}(x)|)$, $k=1,2,...$, with certain positive constants d_k. (17)

Finally, we obtain from the representation of $r_n(x)$ that

$$|r_n^{(k)}(x)|\leqslant 2(2\pi n)^k\sum_{\nu=1}^{\infty}\nu^kb_n^\nu\leqslant 2(2\pi n)^k\frac{b_n}{(1-b_n)^{k+1}}\tilde{d}_k,\ k=1,2,...,$$

where

$$b_n=\exp(-2\pi^2n^2\sigma_n^2)<\frac{m!}{(2\pi^2n^2\sigma_n^2)^m}, \quad \text{for} \quad m=1,2,...$$

with certain positive constants $\tilde{d}_k$ (e.g., $\tilde{d}_0=\tilde{d}_1=1$, $\tilde{d}_2=2$). (18)

Now, assume that the following assumption holds (compare [8]):

$$n^2\sigma_n^2 \geqslant \frac{\log n}{2\pi^2}, \qquad \text{for} \quad n \geqslant 2, \qquad \sigma_1^2 \geqslant \frac{\log 2}{2\pi^2}. \tag{19}$$

This implies $b_n \leqslant 1/n$, $b_1 \leqslant 1/2$, thus by (17) and (18)

$$\sum p_{jn}(x) \leqslant 1 + |r_n(x)| \leqslant 1 + 2\frac{b_n}{1-b_n} \leqslant 3,$$

$$\sum |(\xi_{jn} - x)\ p'_{jn}(x)| = \mu_{2n}(x)/\sigma_n^2 \leqslant 3 + \sigma_n^2 \cdot 2(2\pi n)^2 \frac{2b_n}{(1-b_n)^3} \leqslant 67,$$

and

$$|r_n(x) - r_n(y)| \leqslant \sup |r'_n(t)| \cdot |x-y| \leqslant 2 \cdot 2\pi n \frac{b_n}{(1-b_n)^2} |x-y| \leqslant 16\pi |x-y|.$$

Hence, under assumption (19), the assertion (1*) of Remark 3 holds with $c = 140$ and $\omega^*(h) = 16\pi \cdot h$.

To derive (13) we assume additionally that

$$\sigma_n^2 \leqslant \beta, \qquad \text{for all } n \in \mathbb{N} \text{ and some } \beta > 0. \tag{20}$$

Then $\sigma_n^{2k-2} \leqslant \beta^{k-1}$ and

$$\sigma_n^{2\nu} |r_n^{(2\nu)}(x)| \leqslant 2(2\pi n)^{2\nu} \frac{b_n}{(1-b_n)^{2\nu+1}} \tilde{d}_{2\nu} \cdot \sigma_n^{2\nu} \leqslant 2^{3\nu+2} \nu!\, \tilde{d}_{2\nu}$$

by (18). Hence $\mu_{2k,n}(x) \leqslant d_k^* \sigma_n^2$ with certain positive constants d_k^* by (17) and this yields (13) for all $\sigma > 0$.

Thus, under assumptions (19) and (20), the assertion (1**) of Remark 3 holds with $c = c(\sigma)$ according to Theorem 2 ($c_1 + c_2 = 70$) for all $\sigma > 0$.

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